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A Study of Fractional Line Integral

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Abstract: **In this paper, we study the fractional line integral based on Jumarie type of Riemann-Liouville (R-L) fractional calculus. The major method we used is a new multiplication of fractional analytic functions. On the other hand, two examples are provided to illustrate the fractional line integral. In fact, the fractional line integral is the generalization of line integral in classical calculus.**

Keyword: **Fractional line integral, Jumarie type of R-L fractional calculus, new multiplication, fractional analytic functions.**

I. INTRODUCTION

The history of fractional calculus is almost as long as the development of ordinary calculus theory. As early as 1695, L'Hospital wrote to Leibniz to discuss the fractional derivative of a function. However, it was not until 1819 that Lacroix first proposed the result of a simple function with fractional derivative. Then, after hundreds of years of development, mathematicians such as Euler, Laplace, Fourier, Abel, Riemann, Liouville, Grunwald, Letnikov, Weyl, and Ritz conducted in-depth research, which promoted the development of this discipline.

In the past few decades, fractional calculus has been applied to many fields, such as physics, dynamics, signal processing, robotics, electrical engineering, viscoelasticity, economics, bioengineering, control theory, and electronics [1-10]. However, the definition of fractional derivative is not unique, there are many useful definitions, including Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modified R-L fractional derivative [11-15]. Jumarie modified the definition of R-L fractional derivative with a new formula, and we obtained that the modified fractional derivative of a constant function is zero. Therefore, it is easier to connect fractional calculus with classical calculus by using this definition.

This paper studies the fractional line integral based on Jumarie's modification of R-L fractional calculus. A new multiplication of fractional analytic functions plays an important role in this paper. Moreover, some examples are given to illustrate the fractional line integral. In fact, the fractional line integral is the natural generalization of line integral in ordinary calculus.

II. PRELIMINARIES

First, we introduce the fractional calculus used in this paper and some important properties.

Definition 2.1 ([16])**:** Suppose that $0 < \alpha \leq 1$, and t_0 is a real number. The Jumarie's modified R-L α-fractional derivative is defined by

$$\left({}_{t_0}D_t^{\alpha}\right)[f(t)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t_0}^{t}\frac{f(x)-f(t_0)}{(t-x)^{\alpha}}dx, \tag{1}$$

And the Jumarie's modified R-L α-fractional integral is defined by

$$\left({}_{t_0}I_t^{\alpha}\right)[f(t)] = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{f(x)}{(t-x)^{1-\alpha}}dx\ , \tag{2}$$

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where $\Gamma(w)=\int_0^{\infty} s^{w-1}e^{-s}ds$ is the gamma function defined on $w>0$.

Proposition 2.2 ([17])**:** *If* α,β,t_0,C *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_{t_0}D_t^{\alpha}\right)\left[(t-t_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-t_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{t_0}D_t^{\alpha}\right)[C]=0. \tag{4}$$

In the following, we introduce the definition of fractional analytic function.

Definition 2.3 ([18])**:** Assume that t,t_0, and a_k are real numbers for all k, $t_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(t^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}$ on some open interval containing t_0, then we say that $f_\alpha(t^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Definition 2.4 ([19])**:** If $0<\alpha\leq 1$, and t_0 is a real number. Let $f_\alpha(t^\alpha)$ and $g_\alpha(t^\alpha)$ be two α-fractional analytic functions defined on an interval containing t_0 ,

$$f_\alpha(t^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}, \tag{5}$$

$$g_\alpha(t^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}. \tag{6}$$

Then

$$\begin{aligned}&f_\alpha(t^\alpha)\otimes g_\alpha(t^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}\otimes\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}\\&=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(t-t_0)^{k\alpha}.\end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned}&f_\alpha(t^\alpha)\otimes g_\alpha(t^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}\otimes\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}.\end{aligned} \tag{8}$$

Definition 2.5 ([20])**:** If $0<\alpha\leq 1$, and r is any real number. Let $f_\alpha(x^\alpha)$ be a α-fractional analytic function. Then the α-fractional analytic function $f_\alpha(x^\alpha)^{\otimes r}$ is defined by

$$f_\alpha(x^\alpha)^{\otimes r}=E_\alpha\left(r\cdot Ln_\alpha\left(f_\alpha(x^\alpha)\right)\right). \tag{9}$$

III. RESULTS AND EXAMPLES

In the following, the definition and the property of fractional line integral are introduced.

Definition 3.1: Let $0<\alpha\leq 1$, $x_\alpha(t^\alpha)$, $y_\alpha(t^\alpha)$, $z_\alpha(t^\alpha)$ be functions defined on $[a,b]$, and be α-fractional analytic at $t=a$. Then C_α given by $\mathbf{r}_\alpha(t^\alpha)=x_\alpha(t^\alpha)\mathbf{i}+y_\alpha(t^\alpha)\mathbf{j}$ is called a plane α-fractional curve. Similarly, if C_α is given by $\mathbf{r}_\alpha(t^\alpha)=x_\alpha(t^\alpha)\mathbf{i}+y_\alpha(t^\alpha)\mathbf{j}+z_\alpha(t^\alpha)\mathbf{k}$, then it is called a space α-fractional curve.

Definition 3.2: Suppose that $0<\alpha\leq 1$. If C_α is a plane α-fractional curve on $[a,b]$ given by $\mathbf{r}_\alpha(t^\alpha)=x_\alpha(t^\alpha)\mathbf{i}+y_\alpha(t^\alpha)\mathbf{j}$, and $f\left(x_\alpha(t^\alpha),y_\alpha(t^\alpha)\right)$ is a function defined on $[a,b]$ and it is α-fractional analytic at $t=a$. Then the α-fractional line integral of f along C_α is defined by

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$$\left(I_{C_\alpha}^{\alpha}\right)\left[f(x_\alpha, y_\alpha)\otimes\left({}_aD_t^{\alpha}\right)[s_\alpha]\right]$$
$$= \left({}_aI_t^{\alpha}\right)\left[f\left(x_\alpha(t^\alpha), y_\alpha(t^\alpha)\right)\otimes\left({}_aD_t^{\alpha}\right)[s_\alpha(t^\alpha)]\right]. \tag{10}$$

Where $s_\alpha(t^\alpha) = \left({}_aI_t^{\alpha}\right)\left[\left[\left(\left({}_aD_t^{\alpha}\right)[x_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_aD_t^{\alpha}\right)[y_\alpha(t^\alpha)]\right)^{\otimes 2}\right]^{\otimes\frac{1}{2}}\right]$ is the arc length function of C_α.

Similarly, if C_α is a space α-fractional curve on $[a,b]$ given by $\mathbf{r}_\alpha(t^\alpha) = x_\alpha(t^\alpha)\mathbf{i} + y_\alpha(t^\alpha)\mathbf{j} + z_\alpha(t^\alpha)\mathbf{k}$, and $f\left(x_\alpha(t^\alpha), y_\alpha(t^\alpha), z_\alpha(t^\alpha)\right)$ is α-fractional analytic at $t = a$. Then the α-fractional line integral of f along C_α is defined by

$$\left(I_{C_\alpha}^{\alpha}\right)\left[f(x_\alpha, y_\alpha, z_\alpha)\otimes\left({}_aD_t^{\alpha}\right)[s_\alpha]\right]$$
$$= \left({}_aI_t^{\alpha}\right)\left[f\left(x_\alpha(t^\alpha), y_\alpha(t^\alpha), z_\alpha(t^\alpha)\right)\otimes\left({}_aD_t^{\alpha}\right)[s_\alpha(t^\alpha)]\right]. \tag{11}$$

Where $s_\alpha(t^\alpha) = \left({}_aI_t^{\alpha}\right)\left[\left[\left(\left({}_aD_t^{\alpha}\right)[x_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_aD_t^{\alpha}\right)[y_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_aD_t^{\alpha}\right)[z_\alpha(t^\alpha)]\right)^{\otimes 2}\right]^{\otimes\frac{1}{2}}\right]$ is the arc length function of C_α.

Theorem 3.3: *Let* $0 < \alpha \leq 1$.

Case 1. If $x_\alpha(t^\alpha), y_\alpha(t^\alpha)$ *and* $f\left(x_\alpha(t^\alpha), y_\alpha(t^\alpha)\right)$ *are* α*-fractional analytic at* $t = a$, *then*

$$\left(I_{C_\alpha}^{\alpha}\right)\left[f(x_\alpha, y_\alpha)\otimes\left({}_aD_t^{\alpha}\right)[s_\alpha]\right]$$
$$= \left({}_aI_t^{\alpha}\right)\left[f\left(x_\alpha(t^\alpha), y_\alpha(t^\alpha)\right)\otimes\left[\left(\left({}_aD_t^{\alpha}\right)[x_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_aD_t^{\alpha}\right)[y_\alpha(t^\alpha)]\right)^{\otimes 2}\right]^{\otimes\frac{1}{2}}\right]. \tag{12}$$

Case 2. If $x_\alpha(t^\alpha), y_\alpha(t^\alpha), z_\alpha(t^\alpha)$ *and* $f\left(x_\alpha(t^\alpha), y_\alpha(t^\alpha), z_\alpha(t^\alpha)\right)$ *are* α*-fractional analytic at* $t = a$, *then*

$$\left(I_{C_\alpha}^{\alpha}\right)\left[f(x_\alpha, y_\alpha, z_\alpha)\otimes\left({}_aD_t^{\alpha}\right)[s_\alpha]\right]$$
$$= \left({}_aI_t^{\alpha}\right)\left[f\left(x_\alpha(t^\alpha), y_\alpha(t^\alpha), z_\alpha(t^\alpha)\right)\otimes\left[\left(\left({}_aD_t^{\alpha}\right)[x_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_aD_t^{\alpha}\right)[y_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_aD_t^{\alpha}\right)[z_\alpha(t^\alpha)]\right)^{\otimes 2}\right]^{\otimes\frac{1}{2}}\right]. \tag{13}$$

Proof *Case 1.*Since $s_\alpha(t^\alpha) = \left({}_aI_t^{\alpha}\right)\left[\left[\left(\left({}_aD_t^{\alpha}\right)[x_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_aD_t^{\alpha}\right)[y_\alpha(t^\alpha)]\right)^{\otimes 2}\right]^{\otimes\frac{1}{2}}\right]$, it follows that

$$\left({}_aD_t^{\alpha}\right)[s_\alpha(t^\alpha)] = \left[\left(\left({}_aD_t^{\alpha}\right)[x_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_aD_t^{\alpha}\right)[y_\alpha(t^\alpha)]\right)^{\otimes 2}\right]^{\otimes\frac{1}{2}}. \tag{14}$$

And hence,

$$\left(I_{C_\alpha}^{\alpha}\right)\left[f(x_\alpha, y_\alpha)\otimes\left({}_aD_t^{\alpha}\right)[s_\alpha]\right] = \left({}_aI_t^{\alpha}\right)\left[f\left(x_\alpha(t^\alpha), y_\alpha(t^\alpha)\right)\otimes\left[\left(\left({}_aD_t^{\alpha}\right)[x_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_aD_t^{\alpha}\right)[y_\alpha(t^\alpha)]\right)^{\otimes 2}\right]^{\otimes\frac{1}{2}}\right].$$

Case2. Since $s_\alpha(t^\alpha) = \left({}_aI_t^{\alpha}\right)\left[\left[\left(x_\alpha(t^\alpha)\right)^{\otimes 2} + \left(y_\alpha(t^\alpha)\right)^{\otimes 2} + \left(z_\alpha(t^\alpha)\right)^{\otimes 2}\right]^{\otimes\frac{1}{2}}\right]$, it follows that

$$\left({}_aD_t^{\alpha}\right)[s_\alpha(t^\alpha)] = \left[\left(\left({}_aD_t^{\alpha}\right)[x_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_aD_t^{\alpha}\right)[y_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_aD_t^{\alpha}\right)[z_\alpha(t^\alpha)]\right)^{\otimes 2}\right]^{\otimes\frac{1}{2}}. \tag{15}$$

Thus,

$$\left(I_{C_\alpha}^{\alpha}\right)\left[f(x_\alpha, y_\alpha, z_\alpha)\otimes\left({}_aD_t^{\alpha}\right)[s_\alpha]\right]$$

$$= \left({}_aI_t^\alpha\right)\left[f\left(x_\alpha(t^\alpha), y_\alpha(t^\alpha), z_\alpha(t^\alpha)\right)\otimes\left[\left(\left({}_aD_t^\alpha\right)[x_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_aD_t^\alpha\right)[y_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_aD_t^\alpha\right)[z_\alpha(t^\alpha)]\right)^{\otimes 2}\right]^{\otimes\frac{1}{2}}\right].$$

Q.e.d.

Finally, two examples are provided to illustrate the fractional line integral.

Example 3.4: If $0 < \alpha \leq 1$, C_α is given by $\mathbf{r}_\alpha(t^\alpha) = x_\alpha(t^\alpha)\mathbf{i} + y_\alpha(t^\alpha)\mathbf{j} = 4\frac{1}{\Gamma(\alpha+1)}t^\alpha\mathbf{i} + 3\frac{1}{\Gamma(\alpha+1)}t^\alpha\mathbf{j}$ $\left(0 \leq \frac{1}{\Gamma(\alpha+1)}t^\alpha \leq 1\right)$, and $f\left(x_\alpha(t^\alpha), y_\alpha(t^\alpha)\right) = \left(x_\alpha(t^\alpha)\right)^{\otimes 2} - \left(y_\alpha(t^\alpha)\right)^{\otimes 2}$. Find the α-fractional line integral of f along C_α, $\left(I_{C_\alpha}^\alpha\right)\left[f(x_\alpha, y_\alpha)\otimes\left({}_0D_t^\alpha\right)[s_\alpha]\right]$.

Solution By Theorem 3.3, the α-fractional line integral of f along C_α

$$\left(I_{C_\alpha}^\alpha\right)\left[f(x_\alpha, y_\alpha)\otimes\left({}_0D_t^\alpha\right)[s_\alpha]\right]$$

$$= \left({}_0I_b^\alpha\right)\left[f\left(x_\alpha(t^\alpha), y_\alpha(t^\alpha)\right)\otimes\left[\left(\left({}_0D_t^\alpha\right)[x_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_0D_t^\alpha\right)[y_\alpha(t^\alpha)]\right)^{\otimes 2}\right]^{\otimes\frac{1}{2}}\right] \quad \left(\text{where } b = (\Gamma(\alpha+1))^{\frac{1}{\alpha}}\right)$$

$$= \left({}_0I_b^\alpha\right)\left[7\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes 2}\otimes[16+9]^{\otimes\frac{1}{2}}\right]$$

$$= 35\cdot\left({}_0I_b^\alpha\right)\left[\frac{2}{\Gamma(2\alpha+1)}t^{2\alpha}\right]$$

$$= 70\cdot\left[\frac{1}{\Gamma(3\alpha+1)}t^{3\alpha}\right]_0^b$$

$$= \frac{70\cdot[\Gamma(\alpha+1)]^3}{\Gamma(3\alpha+1)}. \tag{16}$$

Example 3.5: Let $0 < \alpha \leq 1$, C_α be $\mathbf{r}_\alpha(t^\alpha) = x_\alpha(t^\alpha)\mathbf{i} + y_\alpha(t^\alpha)\mathbf{j} + z_\alpha(t^\alpha)\mathbf{k} = \frac{1}{\Gamma(\alpha+1)}t^\alpha\mathbf{i} + 2\frac{1}{\Gamma(\alpha+1)}t^\alpha\mathbf{j} + \frac{1}{\Gamma(\alpha+1)}t^\alpha\mathbf{k}$ $\left(0 \leq \frac{1}{\Gamma(\alpha+1)}t^\alpha \leq 1\right)$, and $f\left(x_\alpha(t^\alpha), y_\alpha(t^\alpha), z_\alpha(t^\alpha)\right) = \left(x_\alpha(t^\alpha)\right)^{\otimes 2} - y_\alpha(t^\alpha) + 3z_\alpha(t^\alpha)$. Evaluate the α-fractional line integral of f along C_α, $\left(I_{C_\alpha}^\alpha\right)\left[f(x_\alpha, y_\alpha, z_\alpha)\otimes\left({}_0D_t^\alpha\right)[s_\alpha]\right]$.

Solution Using Theorem 3.3 yields the α-fractional line integral of f along C_α

$$\left(I_{C_\alpha}^\alpha\right)\left[f(x_\alpha, y_\alpha, z_\alpha)\otimes\left({}_0D_t^\alpha\right)[s_\alpha]\right]$$

$$= \left({}_0I_p^\alpha\right)\left[f\left(x_\alpha(t^\alpha), y_\alpha(t^\alpha), z_\alpha(t^\alpha)\right)\otimes\left[\left(\left({}_0D_t^\alpha\right)[x_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_0D_t^\alpha\right)[y_\alpha(t^\alpha)]\right)^{\otimes 2} + \left(\left({}_0D_t^\alpha\right)[z_\alpha(t^\alpha)]\right)^{\otimes 2}\right]^{\otimes\frac{1}{2}}\right]$$

$\left(\text{where } p = (\Gamma(\alpha+1))^{\frac{1}{\alpha}}\right)$

$$= \left({}_0I_p^\alpha\right)\left[\left(\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes 2} + \frac{1}{\Gamma(\alpha+1)}t^\alpha\right)\otimes[6]^{\otimes\frac{1}{2}}\right]$$

$$= \sqrt{6}\cdot\left({}_0I_p^\alpha\right)\left[\frac{2}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{1}{\Gamma(\alpha+1)}t^\alpha\right]$$

$$= \sqrt{6}\cdot\left[2\cdot\frac{1}{\Gamma(3\alpha+1)}t^{3\alpha} + \frac{1}{\Gamma(2\alpha+1)}t^{2\alpha}\right]_0^p$$

$$= \sqrt{6}\cdot\left(\frac{2\cdot[\Gamma(\alpha+1)]^3}{\Gamma(3\alpha+1)} + \frac{[\Gamma(\alpha+1)]^2}{\Gamma(2\alpha+1)}\right). \tag{17}$$

IV. CONCLUSION

In this paper, we study the fractional line integral based on Jumarie type of R-L fractional calculus. A new multiplication of fractional analytic functions plays an important role in this article. Some examples are given to illustrate the fractional line integral. And we easily know that the fractional line integral is generalization of line integral in traditional calculus. In the future, we will continue to use fractional line integral to solve some problems in fractional calculus and applied mathematics.

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